MHV one-loop amplitudes in Yang-Mills from generalised unitarity

Andreas Brandhuber and Massimiliano Vincon

Centre for Research in String Theory, Department of Physics,
Queen Mary, University of London,
Mile End Road, London, E1 4NS, United Kingdom
E-mail: `a.brandhuber@qmul.ac.uk`, `m.vincon@qmul.ac.uk`

ABSTRACT: In this letter, we exploit generalised unitarity in order to calculate the cut-constructible part of one-loop amplitudes in non-supersymmetric Yang-Mills theory. In particular, we rederive the n-gluon MHV amplitudes for both the adjacent and non-adjacent gluon helicity configurations from three- and four-particle cuts alone.

Contents

1. Introduction

It is common knowledge that the unitarity method, introduced in [1, 2] and further developed in [3], proved itself to be a powerful as well as elegant tool for computing loop scattering amplitudes (see [4] and references therein for a comprehensive review). In fact, recent years have witnessed impressive achievements in the calculation of two- and higher-loop scattering amplitudes with much of the effort mostly focused on the maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory (MSYM) [5–8]. This is primarily due to the simplicity of the perturbative expansion in the 't Hooft (planar) limit of MYSM suggested by an intriguing duality that relates MSYM at strong coupling to weakly-coupled gravity on $AdS_5 \times S^5$ [9]. A short while ago, this duality was exploited as a different manner to compute amplitudes in MSYM [10] and in the case of four-gluon amplitudes agreement was found with an all-loop order ansatz put forward in [6].

In this letter, we focus on one-loop maximally helicity violating (MHV) amplitudes in pure Yang-Mills theory. These amplitudes are of particular interest as they constitute an example of one-loop n-point scattering amplitudes in QCD, where both external and internal particles are gluons. In pure Yang-Mills the n-gluon one-loop amplitudes may be decomposed as

$$\mathcal{A}_{\text{gluon}}^n = \mathcal{A}_{\mathcal{N}=4}^n - 4\mathcal{A}_{chiral,\mathcal{N}=1}^n + \mathcal{A}_{\text{scalar}}^n \, . \tag{1.1}$$

Although each contribution of (1.1) has been computed for the case of MHV amplitudes using the unitarity method [1, 2], the MHV diagram approach [11–14] and, to some extent,

generalised unitarity [15, 16], an explicit double-check of the last term of (1.1), namely the contribution arising from a complex scalar particle running in the loop, is still lacking for the case of MHV amplitudes with non-adjacent negative-helicity gluons.[1] As we felt obliged to do so, we aim in this letter to rederive the cut-constructible scalar contribution to the n-gluon MHV amplitude by means of the generalised unitarity method [17, 18, 3, 15].

At one loop, generalised unitarity instructs us to cut the amplitude into a product of up to four on-shell tree amplitudes and to replace the propagators connecting the sub-amplitudes by on-shell δ-functions,[2] which put the internal particles on shell. When four propagators are cut (quadruple cut) the momentum integral is completely frozen and the re-sulting product of four tree-amplitudes[3] can be identified directly with coefficients of scalar box functions [15]. One route to obtain the coefficients for the remaining scalar triangle and bubble functions is to use triple cuts and conventional two-particle cuts. An efficient method to extract directly, individual coefficients of specific scalar integral functions using a convenient parametrisation for the cut momenta was presented recently in [26].

For the extraction of triangle and bubble coefficients we want to follow a slightly different approach [16, 24]. Here one considers the triple cut of a one-loop amplitude, which in general has contributions from triangle and box functions. One can in principle subtract off the box contributions using quadruple cuts but strictly speaking this is not needed. The three delta functions do not completely freeze the loop integration, hence we simplify the integrand as much as possible using the three loop momentum constraints where the loop momenta are allowed to take complex values. In the final step the cut integral is lifted back up to a loop integral by replacing the on-shell delta functions by the corresponding propagators. The result contains terms that have the correct cuts in the channel under consideration, and possibly terms with cuts in other channels; the latter terms can be dropped. Considering all possible cuts should then give the complete amplitude. An important comment is in order here. The procedure outlined above also produces linear triangle integral functions (triangle integrals with one loop momentum in the numerator), which, as is well known, can be written as linear combinations of scalar triangle and bubble integrals. Therefore, this method can also produce bubble functions which a priori would require the use of additional two-particle cuts. At this point we do not have a proof that two-particle cuts can be avoided for general amplitudes, but for the examples considered in [16, 24] and in this letter this method produces the correct answers. The examples include the Next-to-MHV one-loop amplitudes with adjacent negative helicity gluons considered in [16], all four-point one-loop amplitudes in pure Yang-Mills considered in [24] and the MHV one-loop amplitudes considered in this letter. Obviously, it would be interesting to study this observation in more detail.

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[1]So far that term has only been calculated using MHV diagrams in [13], while the special case of adjacent negative helicity gluons was first found in [2].

[2]Since the solutions of the momentum constraints can be complex in general we replace a cut propagator by $\delta(l_i^2)$ and not by $\delta^{(+)}(l_i^2)$. Also in the subsequent manipulations of the integrands we allow the loop momenta to be complex.

[3]To be more precise, in general the result is a weighted sum over the two complex solutions of the momentum constraints.

In this paper we focus on the rederivation of the cut-constructible parts of MHV one-loop amplitudes by considering a complex scalar running in the loop. In the case that both negative helicity gluons are adjacent all quadruple cuts vanish and, hence, the answer does not contain box functions. In the case that the negative helicity gluons are not adjacent box functions do contribute and can be determined either directly using quadruple cuts (see [16]) or with the triple cut method outlined above. As a consistency check we have also considered the quadruple cuts in section 3. Therefore, in the following discussion we will concentrate on the triple cuts, which in the case at hand allow us to determine the full cut-constructible part of this class of amplitudes. Explicitly, the non-vanishing triple cuts of the scalar loop contribution to the n-gluon MHV amplitude (see figure 2) take the form:

$$\mathcal{A}^n_{\text{scalar}}\Big|_{\text{cut}} = \sum_{\pm} \int d^4\ell_1\, d^4\ell_2\, d^4\ell_3\, \delta(\ell_1^2)\, \delta(\ell_2^2)\, \delta(\ell_3^2)\, \delta^4(\ell_3 - \ell_1 - Q)\delta^4(\ell_1 - \ell_2 - P) \quad (1.2)$$
$$\times \mathcal{A}_{\text{tree}}(\ell_1, (m_2+1), \ldots, j^-, \ldots, -\ell_2)\mathcal{A}_{\text{tree}}(\ell_2, m_1, -\ell_3)$$
$$\times \mathcal{A}_{\text{tree}}(\ell_3, \ldots, i^-, \ldots, m_2, -\ell_1)\,,$$

where the allowed values of m_1 and m_2 are

$$j + 1 \leq m_1 \leq i - 1, \qquad i + 1 \leq m_2 \leq j - 1\,. \quad (1.3)$$

The tree amplitudes entering the integrand involve two MHV amplitudes with two scalars and one anti-MHV three-point amplitude with two scalars. The $\pm$ in (1.2) refers to the fact that we have a complex scalar running in the loop. Thus, there are two possible helicity configurations, each of which gives rise to the same integrand.

On general grounds, four-dimensional cuts alone suffice to reconstruct the full amplitudes in supersymmetric theories at one loop [1, 2]. However, in theories not protected by supersymmetry, there are additional rational terms which cannot be detected by cuts, unless one decides to work in $D = 4 - 2\epsilon$ dimensions and keep higher orders in ϵ, so that even rational terms develop discontinuities which can be detected by the unitarity method. An example of such an amplitude is the one-loop four-gluon $++++$ amplitude with a complex scalar running in the loop. This amplitude consists of purely rational terms and it was first computed in [19] using a technique based on the technology of four-dimensional heterotic string theory. It was subsequently confirmed and extended to the case of an arbitrary number of positive helicity gluons in [20, 21] and to the case when one of the gluons has opposite helicity from the others [20]. Furthermore, the $++++$ one-loop amplitude was recalculated in [22] by means of two-particle cuts in $D = 4 - 2\epsilon$ dimensions, in [23] where a relationship between one-loop MHV gluon amplitudes of QCD and those of $\mathcal{N} = 4$ SYM was put forward and in [24] using the generalised unitarity method in $D = 4 - 2\epsilon$ dimensions. More recently, there has been a proposal [25] in which it was argued that in a particular regularisation scheme certain Lorentz-violating counterterms provide these missing rational terms. We wish to make it clear that in this letter we shall only work with unitarity cuts in $D = 4$ dimensions, thus considering only the cut-constructible part of the n-gluon MHV amplitude. Hence, all the (cut) loop momenta in this letter are kept in four dimensions until the amplitude has been expressed as a linear combination of integral
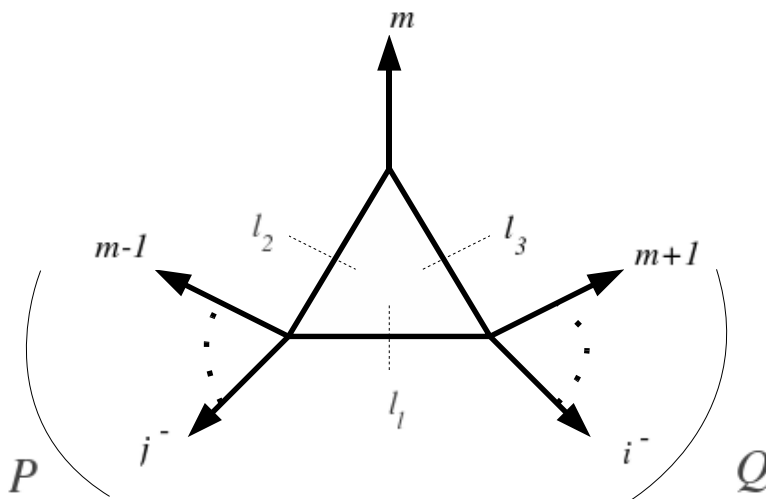
Figure 1: The three-particle cut diagram contributing to the n-gluon amplitude in the case of adjacent negative-helicity gluons.

functions. Only at this stage the dimensional regularisation parameter ϵ is introduced to regularise the divergences of the integral functions.

The cut-constructible part of the MHV one-loop amplitudes in pure Yang-Mills for the special case of adjacent negative helicity gluons has already been calculated in [2] using unitarity whereas the general helicity configuration was dealt with in [13] by means of the MHV diagram method. Note that the rational parts of these amplitudes have been computed analytically in [27, 28] using the powerful method of on-shell recursion relations. The purpose of this letter is to show how generalised unitarity correctly reproduces the cut-constructible parts of the n-gluon amplitudes with less effort than conventional two-particle cuts or the MHV diagram method. We discuss the adjacent negative-helicity case in the next section and the general case in section 3. In section 4 we present our conclusions.

2. MHV one-loop amplitudes: adjacent negative-helicity gluons

In this section we show how generalised unitarity may be used to compute the n-point MHV one-loop amplitude in pure Yang-Mills for the case of adjacent negative-helicity gluons.

Let us consider the triple-cut diagram depicted in figure 1, where we choose all momenta to be outgoing. There are two such diagrams, which are obtained by flipping all the internal helicities of the scalar particles running in the loop. Without loss of generality we set $i = 1$ and $j = 2$ throughout this section. Note that in this case the range of m is $3 \leq m \leq n$. Furthermore, in the adjacent case all quadruple cuts vanish and, hence, no box functions appear in the amplitude.

The triple cut[4] of the n-point amplitude is obtained by sewing three tree-level amplitudes. Ignoring factors of i and 2π, the product of the tree amplitudes appearing in the

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[4]For a short summary of conventions, see the appendices.

triple cut (1.2) is

$$\frac{[m\ell_2][m\ell_3]}{[\ell_2\ell_3]} \times \frac{\langle 1\ell_1\rangle^2\langle 1\ell_3\rangle^2}{\langle(m+1)(m+2)\rangle\ldots\langle n1\rangle\langle 1\ell_1\rangle\langle\ell_1\ell_3\rangle\langle\ell_3(m+1)\rangle} \times$$
$$\frac{\langle 2\ell_1\rangle^2\langle 2\ell_2\rangle^2}{\langle 23\rangle\ldots\langle(m-2)(m-1)\rangle\langle(m-1)\ell_2\rangle\langle\ell_2\ell_1\rangle\langle\ell_1 2\rangle} \ . \tag{2.1}$$

Thus (1.2) together with (2.1) gives

$$\mathcal{A}^n_{\text{scalar}}\Big|_{\text{cut}}$$
$$= 2iA_{\text{tree}}\int\frac{d^4\ell_2\prod_{i=1}^3\delta(l_i^2)}{(2\pi)^4}\frac{\langle 2\,\ell_2\rangle^2\,\langle\ell_1\,2\rangle\,\langle 1\,\ell_1\rangle\,\langle 1\,\ell_3\rangle^2\,\langle(m-1)\,m\rangle\,\langle m(m+1)\rangle[m\,\ell_2]\,[\ell_3\,m]}{\langle 1\,2\rangle^3\,\langle(m-1)\,\ell_2\rangle\,\langle\ell_2\,\ell_1\rangle\,\langle\ell_3\,(m+1)\rangle\,\langle\ell_1\,\ell_3\rangle\,[\ell_2\,\ell_3]}$$
$$= 2i\,A_{\text{tree}}\int\frac{d^4\ell_2}{(2\pi)^4}\,\frac{\langle 2\,|\ell_2|m]\,\langle 1\,m\rangle\,\langle 1\,|P\ell_2|2\rangle\,\langle 2\,|P\ell_2|1\rangle\,[1\,2]^3}{2^5\,(1\cdot 2)^3\,(\ell_1\cdot\ell_2)^2\,\ell_1^2\,\ell_2^2\,\ell_3^2}\Big|_{\text{cut}} \ , \tag{2.2}$$

where in the second line of (2.2) we have factored out the MHV tree level amplitude and cancelled certain spinor brackets in the numerator and denominator of (2.1). In order to arrive at the last line of (2.2) we have used the fact that the holomorphic spinors of the momenta appearing in the anti-MHV three-point amplitude are proportional to each other, i.e. $\lambda_m\propto\lambda_{\ell_2}\propto\lambda_{\ell_3}$. The factor of two accounts for the fact that we have already summed over the two possible internal helicities. Finally, the δ-functions have been replaced by full propagators and the three-particle phase-space integral has been promoted to an unrestricted loop integral. The symbol $|_{\text{cut}}$ indicates that this replacement is only valid in the channel defined by a given triple-cut.

Let us clarify some notations. We define the general external momenta k_p as $k_p := p$. Also, we define

$$P := q_{j,m-1}, \qquad Q := q_{m+1,i}\,, \tag{2.3}$$

where $q_{p_i,p_j} := \sum_{l=p_i}^{p_j}k_l$. We set $i = 1$ and $j = 2$ for the adjacent case.

Converting (2.2) into Dirac traces yields the following integrand:

$$\frac{\text{tr}_+(\cancel{1}\,\cancel{2}\,\cancel{P}\,\cancel{\ell}_2)\,\text{tr}_+(\cancel{1}\,\cancel{2}\,\cancel{\ell}_2\,\cancel{m})\,\text{tr}_+(\cancel{2}\,\cancel{1}\,\cancel{\ell}_2\,\cancel{P})}{2^5\,(1\cdot 2)^3\,(\ell_1\cdot\ell_2)^2} \ . \tag{2.4}$$

Thus, the task reduces to computing the three-index tensor integral

$$\mathcal{I}^{\mu\nu\rho}(m,P,Q) = \int\frac{d^4\ell_2}{(2\pi)^4}\,\frac{\ell_2^\mu\,\ell_2^\nu\,\ell_2^\rho}{\ell_1^2\,\ell_2^2\,\ell_3^2}\,, \tag{2.5}$$

which may be done by standard Passarino-Veltman (PV) integral reduction [29]. Details of the calculation can be found in appendix B.

The result of the PV reduction has to be inserted into (2.1). Doing so yields a series of terms of which, after some manipulations, only the following two remain:

$$A_1 = -\frac{A_{\text{tree}}}{(t_1^{[2]})^2}\frac{1}{6}\frac{\left[I_2(P^2) - I_2(Q^2)\right]}{(Q^2 - P^2)^2}(1\,2\,Q\,m)^2\,, \tag{2.6}$$

$$A_2 = \frac{A_{\text{tree}}}{(t_1^{[3]})^3} \frac{1}{3} \frac{\left[I_2(P^2) - I_2(Q^2)\right]}{(Q^2 - P^2)^3} (1\,2\,Q\,m)^2 (1\,2\,m\,Q)\,, \tag{2.7}$$

where $t_i^{[k]} := (p_i + p_{i+1} + \cdots + p_{i+k-1})^2$ are sums of color-adjacent momenta and the I_2 functions are the scalar bubble functions as defined in appendix A. In obtaining (2.6) and (2.7), we made use of the fact that momentum conservation dictates that on the triple-cut $(\ell_1 \cdot \ell_2)^2 = 4/P^4$ and $(m \cdot Q) = -(m \cdot P) = -(1/2)(Q^2 - P^2)$. Also, in order to make the formulas more compact, we introduced the notation $(a_1\,a_2\,a_3\,a_4) := \mathrm{tr}_+(\slashed{a}_1\,\slashed{a}_2\,\slashed{a}_3\,\slashed{a}_4)$, which we will use throughout the rest of the paper.

In (2.6) and (2.7) the combinations $\left[I_2(P^2) - I_2(Q^2)\right]/((Q^2 - P^2)^{(r)})$ appear, which are ϵ-dependent triangle functions expressed as differences of two bubble functions (defined in section A.1). For convenience we choose to write them as

$$T_\epsilon^{(r)}(m, P, Q) := \frac{1}{\epsilon} \frac{(-P^2)^{-\epsilon} - (-Q^2)^{-\epsilon}}{(Q^2 - P^2)^r}\,, \tag{2.8}$$

where r is a positive integer and the momenta on which $T^{(r)}$ depends satisfy $m + P + Q = 0$.

As mentioned in the Introduction, we are working in $D = 4$ dimensions so that we really should take the $\epsilon \to 0$ limit of (2.8). For $P^2 \neq 0$ and $Q^2 \neq 0$ we define the finite, ϵ-independent triangle function,

$$T^{(r)}(m, P, Q) := \frac{\log(Q^2/P^2)}{(Q^2 - P^2)^r}\,. \tag{2.9}$$

In the event of the vanishing of either of the kinematic invariants, (2.8) gives rise to infrared-divergent terms since one of the numerator terms in (2.8) vanishes. There are two possibilities:

- $P = k_2$ with $P^2 = 0$,

- $Q = k_1$ with $Q^2 = 0$.

Finally, the amplitude takes the following form:

$$\mathcal{A}_n^{\text{scalar}} = \mathcal{A}_{\text{poles}} + \mathcal{A}_1 + \mathcal{A}_2\,, \tag{2.10}$$

where

$$\mathcal{A}_{\text{poles}} = -\frac{i}{6} \mathcal{A}_{\text{tree}} \frac{1}{\epsilon} \left[(-t_2^{[2]})^{-\epsilon} + (-t_n^{[2]})^{-\epsilon}\right]\,, \tag{2.11}$$

$$\mathcal{A}_1 = -\frac{2i}{6} \mathcal{A}_{\text{tree}} \frac{1}{(t_1^{[2]})^2} \sum_{m=4}^{n-1} \left[(1\,2\,P\,m)^2\right] T^{(2)}(m, P, Q)\,,$$

$$\mathcal{A}_2 = -\frac{2i}{3} \mathcal{A}_{\text{tree}} \frac{1}{(t_1^{[2]})^3} \sum_{m=4}^{n-1} \left[(1\,2\,P\,m)^2 (1\,2\,m\,P)\right] T^{(3)}(m, P, Q)\,,$$

where we used $t_i^{[k]} := (p_i + p_{i+1} + \cdots + p_{i+k-1})^2$ and the triangle functions introduced in (2.9).

Equation (2.11),[5] which gives the cut-constructible part of the n-point one-loop scattering amplitudes with two adjacent gluons of negative helicity, agrees with the amplitudes found in [2] using conventional unitarity and with the amplitude found in [13] using MHV diagrams.

3. MHV one-loop amplitudes: non-adjacent negative-helicity gluons

The case in which the two negative-helicity gluons are non-adjacent is more involved. Fortunately, the calculation turns out to be more straightforward than expected, since some of the algebraic manipulations involved can be related to manipulations appearing in the hMHV diagram calculation of the same amplitudes [13].

As in the adjacent case, our starting expression is (1.2). A direct, brute force calculation yields rather unpleasant four-tensor box integrals. However, we do not follow this approach as it would spoil our goal to show the simplicity of the generalised unitarity method. Instead, by using momentum conservation arguments to eliminate ℓ_3 from (1.2), we arrive at a more elegant and manageable expression for the amplitude given by

$$
\mathcal{A}^n_{\text{scalar}}\Big|_{\text{cut}} = -\frac{2iA_{\text{tree}}}{\langle ij\rangle^4}\int\frac{d^4\ell_2}{(2\pi)^4}\frac{\langle j\,\ell_1\rangle^2\langle j\,\ell_2\rangle^2\langle i\,\ell_1\rangle^2\langle i\,\ell_2\rangle^2\langle m_2\,(m_2+1)\rangle\langle(m_1-1)\,m_1\rangle[\ell_2\,m_1]}{\ell_1^2\,\ell_2^2\,\ell_3^2\,\langle\ell_1\,(m_2+1)\rangle\langle(m_1-1)\,\ell_2\rangle\langle m_2\,\ell_1\rangle\langle\ell_1\,\ell_2\rangle^2}\Big|_{\text{cut}},
\tag{3.1}
$$

where in deriving (3.1) we made use of the fact that on the cut

$$
\begin{aligned}
\lambda_{\ell_2} &= \alpha\,\lambda_{m_1}\,,\\
\lambda_{\ell_3} &= \beta\,\lambda_{m_1}\,,\\
\tilde{\lambda}_{\ell_2} &= \frac{1}{\alpha}\tilde{\lambda}_{m_1} + \frac{\beta}{\alpha}\tilde{\lambda}_{\ell_3}\,,
\end{aligned}
\tag{3.2}
$$

for some complex α and β with λ and $\tilde{\lambda}$ holomorphic and antiholomorphic spinors of negative and positive helicity respectively.

In order to reduce the hexagon integral (3.1) to a linear combination of box and triangle integrals, we notice that multiplying and dividing (3.1) by $\langle\ell_2\,m_1\rangle$ allows us to write the integrands,[6] after applying the Schouten identity twice, as a sum of four terms

$$
\mathcal{C}(m_2+1,\,m_1) - \mathcal{C}(m_2+1,\,m_1-1) - \mathcal{C}(m_2,\,m_1) + \mathcal{C}(m_2,\,m_1-1)\,,
\tag{3.3}
$$

where

$$
\mathcal{C}(a,b) := \frac{\langle j\,\ell_1\rangle^2\langle j\,\ell_2\rangle\langle i\,\ell_1\rangle\langle i\,\ell_2\rangle^2}{\langle\ell_1\,\ell_2\rangle^2\langle ij\rangle^4}\cdot\frac{\langle i\,a\rangle\langle j\,b\rangle}{\langle\ell_1\,a\rangle\langle\ell_2\,b\rangle}\,.
\tag{3.4}
$$

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[5]Notify that in the notation of [2, 13] $q_{m,1} = -P$. Also, we dropped an overall, ϵ-dependent factor c_Γ [2] and did not make the symmetry properties of the amplitude under the exchange of the gluons $1 \leftrightarrow 2$ manifest in writing our result, thus explaining a factor of two compared to [2, 13].

[6]The reader might argue, in view of (3.2), that $\langle\ell_2 m_1\rangle$ is zero which entails that we are effectively multiplying (3.1) by $\frac{0}{0}$. However, at this point we are off-shell as we have *uplifted* the cut integral to a Feynman integral by replacing on-shell δ-functions by full Feynman propagators.
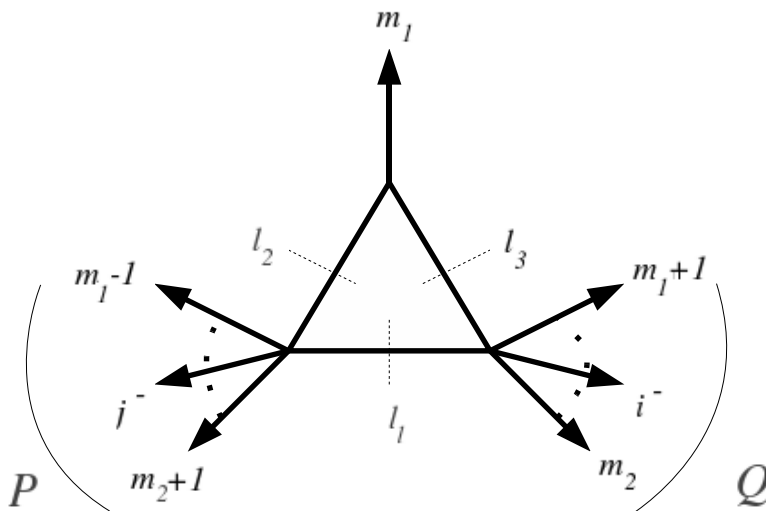
Figure 2: One of the two possible triple cut diagrams contributing to the n-gluon amplitude in the general case. The other triple cut diagram is obtained by swapping i and j through the replacements $m_1 - 1 \to m_1$ and $m_2 \leftrightarrow m_1$.

Therefore, we find

$$\mathcal{A}_{\text{scalar}}^n\Big|_{\text{cut}} = 2i\, A_{\text{tree}} \left[\int \frac{d^4\ell_2}{(2\pi)^4} \frac{1}{\ell_1^2 \ell_2^2} - \int \frac{d^4\ell_2}{(2\pi)^4} \frac{1}{\ell_1^2 \ell_3^2} \right] \sum_{a,b} \mathcal{C}(a,b)\Big|_{\text{cut}}, \qquad (3.5)$$

where the sum stands for the sum of four terms (with signs) in (3.3).

One of the triple cuts contributing to the amplitude may be seen in figure 2 where we defined $P := q_{m_2+1,m_1-1}$ and $Q := q_{m_1+1,m_2}$. Our choice for the momentum flow explains why we find the $\mathcal{C}$ coefficients with $a \leftrightarrow b$ compared to [13].

Although the calculation carried out in [13] is conceptually different from the one we are performing here, we can nevertheless make use of formula (B.16) in that paper, which gives a rather convenient expression for $\mathcal{C}$:

$$-\mathcal{C}(a,b) = \frac{(i\,j\,\ell_1\,\ell_2)(i\,j\,\ell_2\,\ell_1)(i\,j\,\ell_1\,a)(i\,j\,b\,\ell_2)}{2^8(i\cdot j)^4(\ell_1\cdot\ell_2)^2(\ell_1\cdot a)(\ell_2\cdot b)} \qquad (3.6)$$

$$= \frac{1}{2^8(i\cdot j)^4}(\mathcal{H}_1 + \ldots + \mathcal{H}_4)\,,$$

where the $\mathcal{H}_i$ are given by

$$\mathcal{H}_1 := \frac{(i\,j\,b\,a)(i\,j\,\ell_l\,P)(i\,j\,P\,\ell_1)(i\,j\,\ell_1\,a)}{(\ell_1\cdot\ell_2)^2(a\cdot b)(\ell_1\cdot a)} \qquad (3.7)$$

$$- \frac{(i\,j\,b\,a)(i\,j\,P\,\ell_2)(i\,j\,\ell_2\,P)(i\,j\,\ell_2\,b)}{(\ell_1\cdot\ell_2)^2(a\cdot b)(\ell_2\cdot b)}\,,$$

$$\mathcal{H}_2 := -\frac{(i\,j\,a\,b)(i\,j\,b\,a)(i\,j\,P\,\ell_1)(i\,j\,\ell_1\,a)}{(\ell_1\cdot\ell_2)(a\cdot b)^2(\ell_1\cdot a)} \qquad (3.8)$$

$$- \frac{(i\,j\,a\,b)(i\,j\,b\,a)(i\,j\,\ell_2\,P)(i\,j\,\ell_2\,b)}{(\ell_1\cdot\ell_2)(a\cdot b)^2(\ell_2\cdot b)}\,,$$

$$\mathcal{H}_3 := -\frac{(i\,j\,a\,b)^2(i\,j\,b\,a)(i\,j\,\ell_1\,a)}{(a\cdot b)^3(\ell_1\cdot a)} \tag{3.9}$$

$$+\frac{(i\,j\,a\,b)^2(i\,j\,b\,a)(ij\ell_2 b)}{(a\cdot b)^3(\ell_2\cdot b)}\,,$$

$$\mathcal{H}_4 := -\frac{(i\,j\,a\,b)^2(i\,j\,b\,a)^2(b\,P\,\ell_1\,a)}{4(a\cdot b)^4(\ell_1\cdot a)(\ell_2\cdot b)}\,. \tag{3.10}$$

Thus, we produce, in ascending order, linear box integrals and linear, two-tensor and three-tensor triangle integrals. We focus first on the triangle integral contributions.

Substituting for a and b in the expressions for $\mathcal{H}$ and keeping only those terms that actually contribute to the particular triple cut depicted in figure 2 yields combinations of differences of traces. In order to express our result in a more compact fashion, we find it useful to define the following quantities:

$$A^{ij}_{m_1 m_2} := \frac{(i\,j\,m_1\,m_2+1)}{(m_1\cdot(m_2+1))} - \frac{(i\,j\,m_1\,m_2)}{(m_1\cdot m_2)}\,, \tag{3.11}$$

$$S^{ij}_{m_1 m_2} := \frac{(i\,j\,m_1\,m_2+1)(i\,j\,m_2+1\,m_1)}{(m_1\cdot(m_2+1))^2} - \frac{(i\,j\,m_1\,m_2)(i\,j\,m_2\,m_1)}{(m_1\cdot m_2)}\,, \tag{3.12}$$

$$I^{ij}_{m_1 m_2} := \frac{(i\,j\,m_1\,m_2+1)(i\,j\,m_2+1\,m_1)^2}{(m_1\cdot(m_2+1))^3} - \frac{(i\,j\,m_1\,m_2)(i\,j\,m_2\,m_1)^2}{(m_1\cdot m_2)^3}\,, \tag{3.13}$$

which exhibit the following symmetry properties

$$A^{ij}_{m_1 m_2} = -A^{ji}_{m_1 m_2}\,, \quad S^{ij}_{m_1 m_2} = S^{ji}_{m_1 m_2}\,. \tag{3.14}$$

The only integrals that survive from (3.5) are the ones with the correct triple cut, i.e. those integrals that have all three propagators that are cut in figure 2. Hence, many of the triangle integrals can be neglected[7] and after the dust has settled we are left with:

$$\mathcal{H}_1 = A^{ij}_{m_1 m_2}\frac{(i\,j\,P\,\ell_2)(i\,j\,\ell_2\,P)(i\,j\,\ell_2\,m_1)}{2^8(i\cdot j)^4(\ell_1\cdot\ell_2)^2(\ell_2\cdot m_1)}\,, \tag{3.15}$$

$$\mathcal{H}_2 = S^{ij}_{m_1 m_2}\frac{(i\,j\,\ell_2\,P)(i\,j\,\ell_2\,m_1)}{2^8(i\cdot j)^4(\ell_1\cdot\ell_2)(\ell_2\cdot m_1)}\,, \tag{3.16}$$

$$\mathcal{H}_3 = I^{ij}_{m_1 m_2}\frac{(i\,j\,\ell_2\,m_1)}{2^8(i\cdot j)^4(\ell_2\cdot m_1)}\,. \tag{3.17}$$

Before we present the complete amplitude, we wish to inspect the coefficient of the box function depicted in figure 3 and compare it with the results found in [13] using MHV diagrams and in [16] using quadruple cuts. The crucial term in the function $\mathcal{C}(a,b)$ that enters the triple cut (3.5) of the amplitude and gives rise to a triple cut of a box function is:

$$-\frac{1}{2^8(i\cdot j)^4}\mathcal{H}_4 = \left[\frac{(i\,j\,m_2\,m_1)^2(i\,j\,m_1\,m_2)^2}{2^8(i\cdot j)^4(m_2\cdot m_1)^4}\right]\frac{(m_1\,P\,\ell_1\,m_2)}{4(l_1\cdot m_2)(l_2\cdot m_1)}\,, \tag{3.18}$$

---

[7]One consequence of these considerations is that the first integral on the right hand side of (3.5) can be ignored altogether.
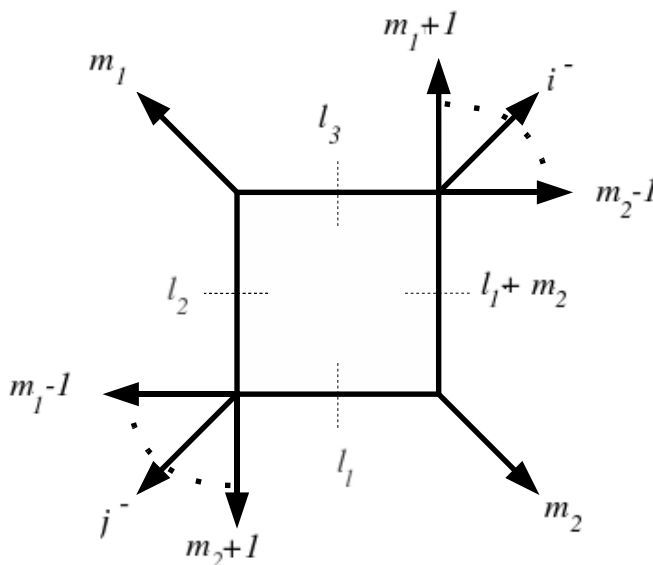
Figure 3: A box functions contributing to the n-gluon MHV amplitude in the general case.

which may be written more compactly as

$$-\frac{1}{2^8(i\cdot j)^4}\mathcal{H}_4 = \frac{1}{4}\left[b^{ij}_{m_1m_2}\right]^2 \frac{(m_1\,P\,\ell_1\,m_2)}{(l_1\cdot m_2)(l_2\cdot m_1)}\,, \tag{3.19}$$

in terms of the coefficient of the box integral function appearing in the one-loop $\mathcal{N}=1$ MHV amplitude with the same helicity configuration computed in [2]

$$b^{ij}_{m_1m_2} := -\frac{1}{8}\frac{(i\,j\,m_2\,m_1)(i\,j\,m_1\,m_2)}{(i\cdot j)^2\,(m_1\cdot m_2)^2}\,. \tag{3.20}$$

Note that (3.19) gives rise to a linear two-mass easy box integral whose PV reduction has been performed in appendix A.3. Inserting the result of this PV reduction into (3.19) reproduces the correct coefficient of the box function. A brief comment is in order here. In the final result [13] only the finite part $B(s,t,P^2,Q^2)$ of the two-mass easy box function appears (as defined e.g. in eq. (4.7) of [13]). We have checked that this is indeed the case and is due to the presence of scalar triangle functions in the PV reduction of appendix A.3 which precisely cancel the IR divergences of the scalar box function $I_4[1]$ once all triple-cut channels are taken into account.

We can now present the complete result[8] for the one-loop n-gluon MHV amplitude (1.2)

---

[8] We have already multiplied by a factor of 2 due to the two scalar helicity configurations running in the loop

reconstructed using the generalised unitarity method:

$$
\mathcal{A}^n_{\text{scalar}} = 2i\,\mathcal{A}_{\text{tree}} \left\{ \sum_{m_1=j+1}^{i-1} \sum_{m_2=i}^{j-1} \frac{1}{2} [b^{ij}_{m_1 m_2}]^2 F\left(t^{[m_2-m_1]}_{m_1}, t^{[m_2-m_1-1]}_{m_1+1}, P, Q \right) \right. \tag{3.21}
$$

$$
+ \left(\frac{8}{3} \sum_{m_1=j+1}^{i-1} \sum_{m_2=i}^{j-1} \left[\mathcal{A}^{ij}_{m_1 m_2} T^{(3)}(m_1, P, Q) + (i \cdot j) \tilde{\mathcal{A}}^{ij}_{m_1 m_2} T^{(2)}(m_1, P, Q) \right] \right.
$$

$$
+ 2 \sum_{m_1=j+1}^{i-1} \sum_{m_2=i}^{j-1} \left[\mathcal{S}^{ij}_{m_1 m_2} T^{(2)}(m_1, P, Q) - \mathcal{I}^{ij}_{m_1 m_2} T(m_1, P, Q) \right]
$$

$$
\left. \left. + (i \leftrightarrow j) \right) \right\},
$$

where we have introduced for convenience the following quantities:

$$
\mathcal{A}^{ij}_{m_1 m_2} := -2^{-8}(i \cdot j)^{-4} A^{ij}_{m_1 m_2} \left[(i\,j\,m_1\,Q)(i\,j\,Q\,m_1)^2 \right] , \tag{3.22}
$$

$$
\tilde{\mathcal{A}}^{ij}_{m_1 m_2} := -2^{-8}(i \cdot j)^{-4} A^{ij}_{m_1 m_2} \left[(i\,j\,Q\,m_1)^2 \right] , \tag{3.23}
$$

$$
\mathcal{S}^{ij}_{m_1 m_2} := 2^{-8}(i \cdot j)^{-4} S^{ij}_{m_1 m_2} \left[(i\,j\,Q\,m_1)^2 \right] , \tag{3.24}
$$

$$
\mathcal{I}^{ij}_{m_1 m_2} := 2^{-8}(i \cdot j)^{-4} I^{ij}_{m_1 m_2} \left[(i\,j\,Q\,m_1) \right] . \tag{3.25}
$$

The amplitude (3.21) agrees precisely with the result found in [13]. Once again, in deriving (3.21) we did not make use of the symmetry properties of the amplitude under exchange of the i-th and j-th gluon.

Similarly to the adjacent case, the infrared divergent terms may be extracted from the cases when either P^2 or Q^2 vanishes (see figure 2). The case $Q^2 = 0$ corresponds to $m_1 = i-1$ and $m_2 = i$, while $P^2 = 0$ corresponds to $m_1 = j+1$ and $m_2 = j-1$. Hence,

$$
T^{(r)}(p, P, Q) \rightarrow (-)^r \frac{1}{\epsilon} \frac{(-t^{[2]}_{i-1})^{-\epsilon}}{(t^{[2]}_{i-1})^r}, \quad Q^2 \rightarrow 0, \tag{3.26}
$$

$$
T^{(r)}(p, P, Q) \rightarrow -\frac{1}{\epsilon} \frac{(-t^{[2]}_{j})^{-\epsilon}}{(t^{[2]}_{j})^r}, \qquad P^2 \rightarrow 0. \tag{3.27}
$$

Thus, we find the following infrared-divergent terms for $Q^2 = 0$:

$$
-\frac{1}{2\,\epsilon} \cdot (-t^{[2]}_{i-1})^{-\epsilon} 4(i \cdot j) \frac{(i\,j\,i-1\,i+1)}{((i+1) \cdot (i-1))} \tag{3.28}
$$

$$
\cdot \left[\frac{8}{3}(i \cdot j)^2 - 2 \frac{(i\,j\,i+1\,i-1)}{((i+1) \cdot (i-1))(i \cdot j)} + \frac{(i\,j\,i+1\,i-1)(i\,j\,i-1\,i+1)}{((i+1) \cdot (i-1))^2} \right] .
$$

Similarly, we find for $P^2 = 0$ the following:

$$
-\frac{1}{2\,\epsilon} \cdot (-t^{[2]}_{j})^{-\epsilon} 4(i \cdot j) \frac{(i\,j\,j-1\,j+1)}{((j+1) \cdot (j-1))} \tag{3.29}
$$

$$
\cdot \left[\frac{8}{3}(i \cdot j)^2 - 2 \frac{(i\,j\,j+1\,j-1)}{((j+1) \cdot (j-1))(i \cdot j)} + \frac{(i\,j\,j+1\,j-1)(i\,j\,j-1\,j+1)}{((j+1) \cdot (j-1))^2} \right] .
$$

4. Conclusions

We have shown how triple cuts correctly reproduce the cut-constructible part of the n-gluon one-loop MHV scattering amplitudes in pure Yang-Mills, both for the adjacent and for the general case. An interesting observation of this calculation is that we did not have to make use of two particle cuts. Of course, our result is consistent with two particle cuts since it agrees with the earlier calculation of the same class of amplitudes in [2] and [13] using conventional unitarity and MHV diagram, respectively. This is in line with similar observations made in [16] and [24] where certain classes of amplitudes where obtained from triple cuts (and quadruple cuts) alone. The particular examples are the Next-to-MHV one-loop amplitudes with adjacent negative helicity gluons considered in [16] and all four-point one-loop amplitudes in pure Yang-Mills considered in [24]. Obviously, it would be interesting to investigate these observations further and understand whether this works for general amplitudes.

A first, important step would be to gain knowledge of the one-loop n-gluon next-to maximally helicity violating amplitudes (NMHV), that is amplitudes with three negative helicities. While the purely gluonic 6-, 7- and n-point one-loop $\mathcal{N} = 4$ NMHV amplitudes were computed in [2, 15, 30, 31] using generalised unitarity, 6- and n-point one-loop amplitudes involving adjoint fermions and scalars in $\mathcal{N} = 4$ gauge theory were found in [32, 33]. A different approach was employed in [34] for the 7-gluon amplitudes in $\mathcal{N} = 4$ NMHV, whereby the authors managed to exploit the holomorphic anomaly of unitarity cuts to reconstruct the amplitude by evaluating the action of a certain differential operator on the cut. Furthermore, the holomorphic anomaly was also utilised in [35] to compute the 6-point one-loop $\mathcal{N} = 1$ split-helicity NMHV amplitude, while the remaining 6-point one-loop $\mathcal{N} = 1$ NMHV amplitudes were calculated in [36]. Generalised unitarity provided the n-gluon one-loop $\mathcal{N} = 1$ NMHV amplitude in [16] for the case that the three negative helicity gluons are adjacent. This latter amplitude has been calculated in pure Yang-Mills in [37] using an iterative approach. Finally, the coefficients of bubble and triangle integral functions for non-supersymmetric six-gluon amplitudes were computed in [38].

Let us conclude with some remarks on preliminary investigations of the NMHV case. We have started to investigate a particular class of non-supersymmetric NMHV amplitudes, namely $\mathcal{A}^n_{\mathrm{scalar}}(1^+, \ldots, i^-, j^-, \ldots, k^-, \ldots n^+)$, i.e. amplitudes where the i-th and j-th negative helicity gluons are adjacent and the k-th one is in an arbitrary position. In order to tackle the problem, we start by identifying all possible triple cuts contributing to the amplitude, which may be seen in figure 4. The triple cut drawn in figure 4a poses no new problems (we found structures similar to those appearing in the calculation of the MHV amplitude we investigated in this letter). For the remaining triple cuts in figure 4b, 4c and 4d an additional difficulty arises, since the tree amplitudes appearing in the triple cut (1.2) may be NMHV. Thus, we cannot employ the Parke-Taylor formula for the standard MHV tree amplitudes. In [39], it was shown how tree amplitudes in Yang-Mills theories may be derived by sewing together MHV vertices, suitably continued off-shell (CSW prescription), and connected by scalar bosonic propagators $1/p^2$ (see [40] for a review). This novel diagrammatic approach stemmed from an insight which relates the perturbative expansion of
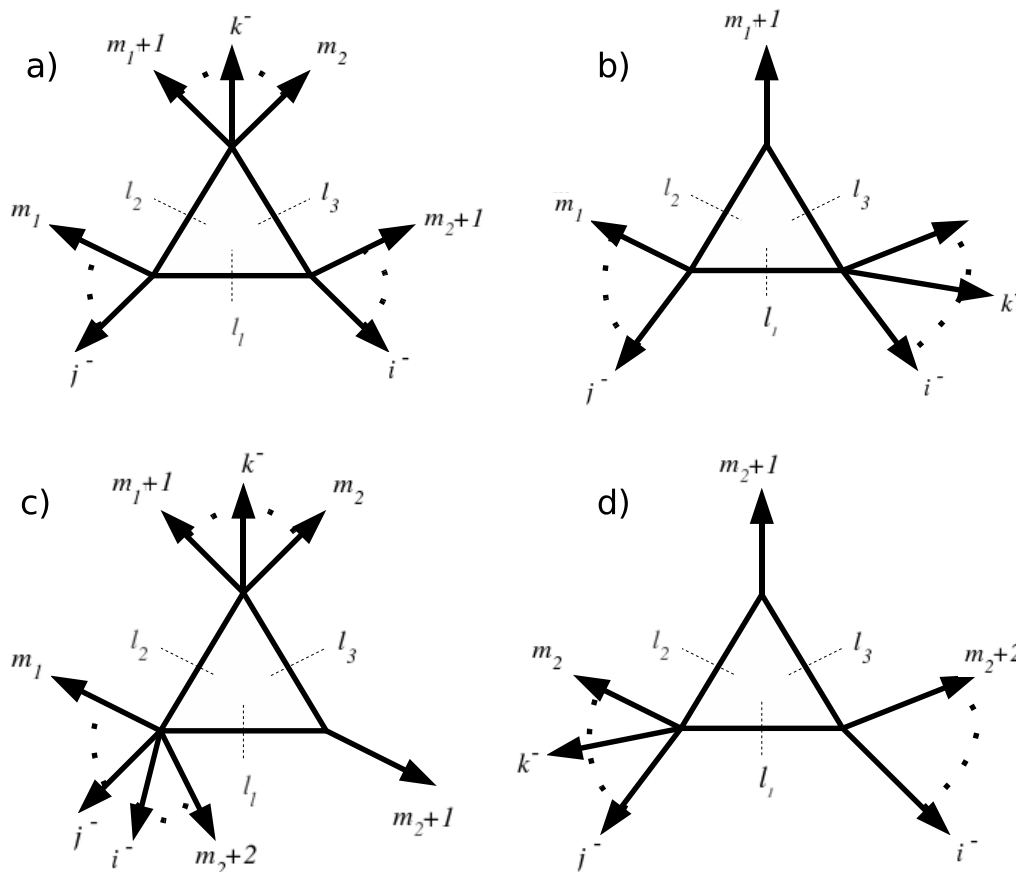
Figure 4: The triple-cut diagrams contributing to the n-gluon one-loop NMHV amplitude.

$\mathcal{N} = 4$ super Yang-Mills theory to D-instanton expansion in the topological B model on super twistor space $CP^{3|4}$ [41]. By applying manipulations similar to those used in this letter, we mostly obtain three-tensor triangle integrals although some more complicated three-tensor pentagon integrals still appear.[9] In a straightforward application of the CSW rules spurious poles arise and it is necessary to use improved formulas for the NMHV tree amplitudes [43] that have only physical poles. We plan to study this case in more detail in the future.

Acknowledgments

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[9]In dealing with the NMHV tree amplitudes, we used the results of [42 – 44].

A. Tensor integrals

In this appendix we define the one-loop integrals[10] encountered in this paper, which were used in performing the PV reductions. Furthermore, we present formulas for the PV reductions of all tensor bubble, triangle and box integrals appearing in our letter. The more complicated three-tensor triangle integral is dealt with separately in appendix B.

A.1 Bubble integrals

A general bubble integral is defined by

$$I_2[P(\ell^\mu)] = -i(4\pi)^2 \int \frac{d^4\ell}{(2\pi)^{4-2\epsilon}} \frac{P(\ell^\mu)}{\ell^2(\ell-K)^2} \,, \tag{A.1}$$

where K is the total outgoing momentum at one side of the bubble and $P(\ell^\mu)$ is some polynomial of the loop momentum ℓ^μ. Evaluation of the scalar bubble integral yields

$$I_2[1] = r_\Gamma \frac{(-K^2)^{-\epsilon}}{\epsilon(1-2\epsilon)} = r_\Gamma \left[\left(\frac{1}{\epsilon} + 2 - \ln(-K^2) \right) + \mathcal{O}(\epsilon) \right], \tag{A.2}$$

where

$$r_\Gamma = \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}. \tag{A.3}$$

Thus, we see that the difference of two scalar bubbles gives rise to (2.9) to $\mathcal{O}(\epsilon^0)$.

The PV reductions of the linear and two-tensor bubble integrals are given by

$$I_2[\ell^\mu] = -\frac{1}{2} I_2[1] K^\mu \,, \tag{A.4}$$

$$I_2[\ell^\mu \ell^\nu] = I_2[1] \left(\frac{1}{3} K^\mu K^\nu - \frac{1}{12} K^2 \eta^{\mu\nu} \right) \,. \tag{A.5}$$

A.2 Triangle integrals

A general tensor triangle integral is defined by

$$I_3[P(\ell^\mu)] = i(4\pi)^2 \int \frac{d^4\ell}{(2\pi)^{4-2\epsilon}} \frac{P(\ell^\mu)}{\ell^2(\ell-K_1)^2(\ell+K_3)^2} \,, \tag{A.6}$$

where the K_i are sums of the momenta k_i of the external gluons at each vertex. We find that the linear and two-tensor two-mass triangle integrals, with momentum assignments as in figure 1, are given by

$$I_3[\ell_2^\mu] = -T^{(1)}(m,P,Q)P^\mu + \dots \,, \tag{A.7}$$

$$I_3[\ell_2^\mu \ell_2^\nu] = \frac{1}{2} T^{(1)}(m,P,Q)P^\mu P^\nu - \frac{1}{2} P^2 T^{(2)}(m,P,Q) \left(P^\mu m^\nu + P^\nu m^\mu \right) + \dots \,. \tag{A.8}$$

The triangle functions $T^{(r)}(m,P,Q)$ have been defined in eq. (2.9) and in the formulas above only those terms have been written down which survive after inserting the tensor integrals in the explicit cut expressions.

---

[10]We follow closely the conventions of [2].

A.3 Box integrals

A general tensor box integral is defined as

$$I_4[P(\ell^\mu)] = -i(4\pi)^2 \int \frac{d^4\ell}{(2\pi)^{4-2\epsilon}} \frac{P(\ell^\mu)}{\ell^2(\ell - K_1)^2(p - K_1 - K_2)^2(\ell + K_4)^2} \qquad (A.9)$$

For the linear box integral with momentum assignments as in figure 3 we find

$$I_4[\ell_1^\mu] = \frac{(m_1 \cdot m_2)\, P^2 I_4[1] - (m_1 \cdot P)\,[I_3 + 2(m_2 \cdot P)\, I_4[1]]}{2\,[(m_1 \cdot m_2)\, P^2 - 2\,(m_2 \cdot P)(m_1 \cdot P)]} P^\mu \qquad (A.10)$$

$$+ \frac{(m_1 \cdot m_2)P^2[I_3 - (m_2 \cdot P)I_4[1]] + (m_1 \cdot P)(m_2 \cdot P)[2\, I_4[1](m_2 \cdot P) - I_3]}{2(m_1 \cdot m_2)\,[(m_1 \cdot m_2)\, P^2 - 2\,(m_2 \cdot P)(m_1 \cdot P)]} m_1^\mu + \dots ,$$

where we are omitting a term proportional to m_2^μ term since it drops out when inserted in (3.19). We refer the interested reader to the appendices I and II of [2] for a more complete discussion of bubble, triangle and box integrals.

B. Passarino-Veltman reduction

In this section we carry out the PV reduction of the three-index tensor two-mass triangle integral, which enters in (2.5) and (3.15)–(3.17) and whose momentum assignments can be found in figure 1:

$$\mathcal{I}^{\mu\nu\rho}(m, P, Q) = \int d^4\ell_2 \frac{\ell_2^\mu\, \ell_2^\nu\, \ell_2^\rho}{\ell_1^2\, \ell_2^2\, \ell_3^2} . \qquad (B.1)$$

The integral (B.1) may be decomposed as

$$\mathcal{I}^{\mu\nu\rho} = a(P^\mu P^\nu P^\rho) + b(P^\mu m^\nu m^\rho + P^\nu m^\mu m^\rho + P^\rho m^\nu m^\mu) + \qquad (B.2)$$
$$c(P^\mu P^\nu m^\rho + P^\mu P^\rho m^\nu + P^\nu P^\rho m^\mu) + d(P^\mu \eta^{\rho\nu} + P^\nu \eta^{\mu\rho} + P^\rho \eta^{\mu\nu}) +$$
$$e(m^\mu \eta^{\nu\rho} + m^\nu \eta^{\mu\rho} + m^\rho \eta^{\nu\mu}) + f(m^\mu m^\nu m^\rho).$$

Taking contractions with all possible momenta then yields

- $P_\mu P_\nu P_\rho$

$$\mathcal{I}_1 = \int \frac{(\ell_2 \cdot P)^3}{\ell_1^2\, \ell_2^2\, \ell_3^2} = a P^8 + 3b[P^2(m \cdot P)^2] + 3c[(m \cdot P)P^4] + \qquad (B.3)$$
$$+ 3d P^4 + 3e[(m \cdot P)P^2] + f(m \cdot P)^3 ,$$

- $P_\mu m_\nu m_\rho$

$$\mathcal{I}_2 = \int \frac{(\ell_2 \cdot P)(m \cdot \ell_2)^2}{\ell_1^2\, \ell_2^2\, \ell_3^2} = a[P^2(m \cdot P)^2] + c(m \cdot P)^3 + 2d(m \cdot P)^2 , \qquad (B.4)$$

- $P_\mu P_\nu m_\rho$

$$\mathcal{I}_3 = \int \frac{(m \cdot \ell_2)(\ell_2 \cdot P)^2}{\ell_1^2\, \ell_2^2\, \ell_3^2} = a[(m \cdot P)P^4] + b(m \cdot P)^3 + 2c[P^2(m \cdot P)^2] + \qquad (B.5)$$
$$3d[P^2(m \cdot P)] + 2e(m \cdot P)^2 ,$$

- $P_\mu \eta_{\nu\rho}$

$$\mathcal{I}_4 = \int \frac{(P \cdot \ell_2)}{\ell_1^2 \, \ell_3^2} = aP^4 + 2b[(m \cdot P)^2] + 3c[P^2(m \cdot P)] + \tag{B.6}$$
$$6dP^2 + 6e(m \cdot P)\,,$$

- $m_\mu \eta_{\nu\rho}$

$$\mathcal{I}_5 = \int \frac{(\ell_2 \cdot m)}{\ell_1^2 \, \ell_3^2} = a[(m \cdot P)P^2] + 2c(m \cdot P)^2 + 6d(m \cdot P)\,, \tag{B.7}$$

- $m_\mu m_\nu m_\rho$

$$\mathcal{I}_6 = \int \frac{(\ell_2 \cdot m)^3}{\ell_1^2 \, \ell_2^2 \, \ell_3^2} = a(m \cdot P)^3\,. \tag{B.8}$$

The integrals take the following values:

$$\mathcal{I}_1 = -\frac{1}{2}(m \cdot P)^2 I_2(Q^2) - \frac{1}{8}P^2 I_3 - \frac{1}{6}(P \cdot Q)^2 I_2(Q^2) + \frac{1}{24}Q^2 P^2 I_2(Q^2) \tag{B.9}$$
$$-\frac{1}{2}(m \cdot P)(P \cdot Q)I_2(Q^2) + \frac{1}{4}(m \cdot P)I_2(Q^2)$$
$$+\frac{1}{8}P^2(P \cdot Q)I_2(Q^2) - \frac{1}{8}P^4 I_2(Q^2)\,,$$
$$\mathcal{I}_2 = -\frac{1}{6}(m \cdot Q)^2 I_2(Q^2) - \frac{1}{8}P^2(m \cdot Q)I_2(Q^2) - \frac{1}{8}P^2(m \cdot P)I_2(P^2)\,,$$
$$\mathcal{I}_3 = \frac{1}{2}(m \cdot P)^2 I_2(Q^2) + \frac{1}{6}(P \cdot Q)^2 I_2(Q^2) + \frac{1}{2}(m \cdot P)(P \cdot Q)I_2(Q^2)$$
$$-\frac{1}{24}Q^2 P^2 I_2(Q^2) - \frac{1}{6}P^4 I_2(P^2) + \frac{1}{24}P^4 I_2(P^2)\,,$$
$$\mathcal{I}_4 = (m \cdot P)I_2(Q^2) + \frac{1}{2}(P \cdot Q)I_2(Q^2)\,,$$
$$\mathcal{I}_5 = \frac{1}{2}(m \cdot Q)I_2(Q^2)\,,$$
$$\mathcal{I}_6 = \frac{1}{6}(m \cdot Q)^2 I_2(Q^2) - \frac{1}{6}(m \cdot P)^2 I_2(P^2)\,,$$

Finally, using Mathematica to carry out the algebraic manipulations, we retrieve the coefficients of the expansion (B.2)

$$a = \frac{I_2(Q^2) - I_2(P^2)}{3Q^2 - 3P^2}\,, \tag{B.10}$$
$$b = \frac{P^4(I_2(P^2 - I_2(Q^2))}{3(P^2 - Q^2)^3}\,,$$
$$c = \frac{P^2(I_2(Q^2) - I_2(P^2))}{6(P^2 - Q^2)^2}\,,$$
$$d = \frac{Q^2 I_2(Q^2) - P^2 I_2(P^2)}{12(P^2 - Q^2)}\,,$$
$$e = \frac{(Q^4 - 2P^2 Q^2)I_2(Q^2) + P^4 I_2(P^2)}{12(P^2 - Q^2)^2}\,,$$

where we chose not to write the f coefficient as one can easily check that the $m_\mu m_\nu m_\rho$ term vanishes once inserted into the appropriate Dirac trace formulas appearing in our calculations. Incidentally, the f coefficient is the only place where the I_3 scalar triangle function appears.

Thus, (B.1) takes the following form:

$$
\int d^4\ell_2 \frac{\ell_2^\mu \ell_2^\nu \ell_2^\rho}{\ell_1^2 \ell_2^2 \ell_3^2} = \frac{I_2(Q^2) - I_2(P^2)}{3Q^2 - 3P^2}(P^\mu P^\nu P^\rho) + \frac{P^4(I_2(P^2) - I_2(Q^2))}{3(P^2 - Q^2)^3}(P^\mu m^\nu m^\rho) \quad \text{(B.11)}
$$
$$
+ \frac{P^2(I_2(Q^2) - I_2(P^2))}{6(P^2 - Q^2)^2}(P^\mu P^\nu m^\rho) + \frac{Q^2 I_2(Q^2) - P^2 I_2(P^2)}{12(P^2 - Q^2)}(P^\mu \eta^{\nu\rho})
$$
$$
+ \frac{(Q^4 - 2P^2 Q^2)I_2(Q^2) + P^4 I_2(P^2)}{12(P^2 - Q^2)^2}(m^\mu \eta^{\nu\rho}).
$$

C. Spinor identities

We list here some spinor identities. The Schouten identity is given by

$$
\langle i\,j \rangle \langle k\,l \rangle = \langle i\,l \rangle \langle k\,j \rangle + \langle i\,k \rangle \langle j\,l \rangle. \quad \text{(C.1)}
$$

Other useful identities are

$$
[i\,j]\,\langle j\,i \rangle = \text{tr}_+(\not{k}_i \not{k}_j) = 2(k_i \cdot k_j), \quad \text{(C.2)}
$$
$$
[i\,j]\,\langle j\,l \rangle [l\,m]\langle m\,i \rangle = \text{tr}_+(\not{k}_i \not{k}_j \not{k}_l \not{k}_m). \quad \text{(C.3)}
$$

In dealing with Dirac traces, we made use of the following identities:

$$
\text{tr}_+(\not{k}_i \not{k}_j \not{k}_l \not{k}_m) = \text{tr}_+(\not{k}_m \not{k}_l \not{k}_j \not{k}_i) = \text{tr}_+(\not{k}_l \not{k}_m \not{k}_i \not{k}_j), \quad \text{(C.4)}
$$
$$
\text{tr}_+(\not{k}_i \not{k}_j \not{k}_l \not{k}_m) = 4(k_i \cdot k_j)(k_l \cdot k_m) - \text{tr}_+(\not{k}_j \not{k}_i \not{k}_l \not{k}_m). \quad \text{(C.5)}
$$

References

[1] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *One loop n point Gauge theory amplitudes, unitarity and collinear limits*, *Nucl. Phys.* **B 425** (1994) 217 [hep-ph/9403226].

[2] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *Fusing Gauge theory tree amplitudes into loop amplitudes*, *Nucl. Phys.* **B 435** (1995) 59 [hep-ph/9409265].

[3] Z. Bern, L.J. Dixon and D.A. Kosower, *One-loop amplitudes for e^+e^- to four partons*, *Nucl. Phys.* **B 513** (1998) 3 [hep-ph/9708239].

[4] Z. Bern, L.J. Dixon and D.A. Kosower, *On-shell methods in perturbative QCD*, *Ann. Phys. (NY)* **322** (2007) 1587 [arXiv:0704.2798].

[5] C. Anastasiou, Z. Bern, L.J. Dixon and D.A. Kosower, *Planar amplitudes in maximally supersymmetric Yang-Mills theory*, *Phys. Rev. Lett.* **91** (2003) 251602 [hep-th/0309040].

[6] Z. Bern, L.J. Dixon and V.A. Smirnov, *Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond*, *Phys. Rev.* **D 72** (2005) 085001 [hep-th/0505205].

[7] Z. Bern, M. Czakon, L.J. Dixon, D.A. Kosower and V.A. Smirnov, *The four-loop planar amplitude and Cusp anomalous dimension in maximally supersymmetric Yang-Mills theory*, *Phys. Rev.* **D 75** (2007) 085010 [`hep-th/0610248`].

[8] Z. Bern, J.J.M. Carrasco, H. Johansson and D.A. Kosower, *Maximally supersymmetric planar Yang-Mills amplitudes at five loops*, *Phys. Rev.* **D 76** (2007) 125020 [`arXiv:0705.1864`].

[9] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [`hep-th/9711200`].

[10] L.F. Alday and J.M. Maldacena, *Gluon scattering amplitudes at strong coupling*, *JHEP* **06** (2007) 064 [`arXiv:0705.0303`].

[11] A. Brandhuber, B.J. Spence and G. Travaglini, *One-loop gauge theory amplitudes in $N = 4$ super Yang-Mills from MHV vertices*, *Nucl. Phys.* **B 706** (2005) 150 [`hep-th/0407214`].

[12] J. Bedford, A. Brandhuber, B.J. Spence and G. Travaglini, *A twistor approach to one-loop amplitudes in $N = 1$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 706** (2005) 100 [`hep-th/0410280`].

[13] J. Bedford, A. Brandhuber, B.J. Spence and G. Travaglini, *Non-supersymmetric loop amplitudes and MHV vertices*, *Nucl. Phys.* **B 712** (2005) 59 [`hep-th/0412108`].

[14] C. Quigley and M. Rozali, *One-loop MHV amplitudes in supersymmetric gauge theories*, *JHEP* **01** (2005) 053 [`hep-th/0410278`].

[15] R. Britto, F. Cachazo and B. Feng, *Generalized unitarity and one-loop amplitudes in $N = 4$ super-Yang-Mills*, *Nucl. Phys.* **B 725** (2005) 275 [`hep-th/0412103`].

[16] S.J. Bidder, N.E.J. Bjerrum-Bohr, D.C. Dunbar and W.B. Perkins, *One-loop gluon scattering amplitudes in theories with $N < 4$ supersymmetries*, *Phys. Lett.* **B 612** (2005) 75 [`hep-th/0502028`].

[17] R.E. Cutkosky, *Singularities and discontinuities of Feynman amplitudes*, *J. Math. Phys.* **1** (1960) 429.

[18] R.J. Eden, P.V. Landshoff, D.I. Olive and J.C. Polkinghorne, *The analytic S-matrix*, Cambridge University Press, Cambridge U.K. (1966).

[19] Z. Bern and D.A. Kosower, *The computation of loop amplitudes in Gauge theories*, *Nucl. Phys.* **B 379** (1992) 451.

[20] G. Mahlon, *Multi-gluon helicity amplitudes involving a quark loop*, *Phys. Rev.* **D 49** (1994) 4438 [`hep-ph/9312276`].

[21] Z. Bern, G. Chalmers, L.J. Dixon and D.A. Kosower, *One loop N gluon amplitudes with maximal helicity violation via collinear limits*, *Phys. Rev. Lett.* **72** (1994) 2134 [`hep-ph/9312333`].

[22] Z. Bern and A.G. Morgan, *Massive loop amplitudes from unitarity*, *Nucl. Phys.* **B 467** (1996) 479 [`hep-ph/9511336`].

[23] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *One-loop self-dual and $N = 4$ super Yang-Mills*, *Phys. Lett.* **B 394** (1997) 105 [`hep-th/9611127`].

[24] A. Brandhuber, S. McNamara, B.J. Spence and G. Travaglini, *Loop amplitudes in pure Yang-Mills from generalised unitarity*, *JHEP* **10** (2005) 011 [`hep-th/0506068`].

[25] A. Brandhuber, B. Spence, G. Travaglini and K. Zoubos, *One-loop MHV rules and pure Yang-Mills*, *JHEP* **07** (2007) 002 [`arXiv:0704.0245`].

[26] D. Forde, *Direct extraction of one-loop integral coefficients*, *Phys. Rev.* **D 75** (2007) 125019 [`arXiv:0704.1835`].

[27] D. Forde and D.A. Kosower, *All-multiplicity one-loop corrections to MHV amplitudes in QCD*, *Phys. Rev.* **D 73** (2006) 061701 [`hep-ph/0509358`].

[28] C.F. Berger, Z. Bern, L.J. Dixon, D. Forde and D.A. Kosower, *All one-loop maximally helicity violating gluonic amplitudes in QCD*, *Phys. Rev.* **D 75** (2007) 016006 [`hep-ph/0607014`].

[29] G. Passarino and M.J.G. Veltman, *One-Loop corrections for e^+e^- annihilation into $\mu^+\mu^-$ in the Weinberg Model*, *Nucl. Phys.* **B 160** (1979) 151.

[30] Z. Bern, V. Del Duca, L.J. Dixon and D.A. Kosower, *All non-maximally-helicity-violating one-loop seven-gluon amplitudes in $N = 4$ super-Yang-Mills theory*, *Phys. Rev.* **D 71** (2005) 045006 [`hep-th/0410224`].

[31] Z. Bern, L.J. Dixon and D.A. Kosower, *All next-to-maximally helicity-violating one-loop gluon amplitudes in $N = 4$ super-Yang-Mills theory*, *Phys. Rev.* **D 72** (2005) 045014 [`hep-th/0412210`].

[32] S.J. Bidder, D.C. Dunbar and W.B. Perkins, *Supersymmetric Ward identities and NMHV amplitudes involving gluinos*, *JHEP* **08** (2005) 055 [`hep-th/0505249`].

[33] K. Risager, S.J. Bidder and W.B. Perkins, *One-loop NMHV amplitudes involving gluinos and scalars in $N = 4$ gauge theory*, *JHEP* **10** (2005) 003 [`hep-th/0507170`].

[34] R. Britto, F. Cachazo and B. Feng, *Computing one-loop amplitudes from the holomorphic anomaly of unitarity cuts*, *Phys. Rev.* **D 71** (2005) 025012 [`hep-th/0410179`].

[35] S.J. Bidder, N.E.J. Bjerrum-Bohr, L.J. Dixon and D.C. Dunbar, *$\mathcal{N} = 1$ supersymmetric one-loop amplitudes and the holomorphic anomaly of unitarity cuts*, *Phys. Lett.* **B 606** (2005) 189 [`hep-th/0410296`].

[36] R. Britto, E. Buchbinder, F. Cachazo and B. Feng, *One-loop amplitudes of gluons in SQCD*, *Phys. Rev.* **D 72** (2005) 065012 [`hep-ph/0503132`].

[37] Z. Bern, N.E.J. Bjerrum-Bohr, D.C. Dunbar and H. Ita, *Recursive calculation of one-loop QCD integral coefficients*, *JHEP* **11** (2005) 027 [`hep-ph/0507019`].

[38] A. Adams, N. Arkani-Hamed, S. Dubovsky, A. Nicolis and R. Rattazzi, *Causality, analyticity and an IR obstruction to UV completion*, *JHEP* **10** (2006) 014 [`hep-th/0602178`].

[39] F. Cachazo, P. Svrček and E. Witten, *MHV vertices and tree amplitudes in gauge theory*, *JHEP* **09** (2004) 006 [`hep-th/0403047`].

[40] A. Brandhuber and G. Travaglini, *Quantum MHV diagrams*, `hep-th/0609011`.

[41] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, *Commun. Math. Phys.* **252** (2004) 189 [`hep-th/0312171`].

[42] G. Georgiou and V.V. Khoze, *Tree amplitudes in gauge theory as scalar MHV diagrams*, *JHEP* **05** (2004) 070 [`hep-th/0404072`].

[43] D.A. Kosower, *Next-to-maximal helicity violating amplitudes in gauge theory*, *Phys. Rev.* **D 71** (2005) 045007 [`hep-th/0406175`].

[44] G. Georgiou, E.W.N. Glover and V.V. Khoze, *Non-MHV tree amplitudes in gauge theory*, *JHEP* **07** (2004) 048 [`hep-th/0407027`].